Propagation in Rectangular Waveguide Filled with Skew Uniaxial Dielectric

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Abstract—**A solution is given for propagation in rectangular waveguide fully loaded with a uniaxial dielectric, with the *c*-axis lying anywhere in the transverse plane. This problem arises in the design of particular traveling-wave masers. By application of the Rayleigh-Ritz method to Berk's variational expression, the problem is reduced to a matrix eigenvalue problem, and in a form suitable for direct evaluation on a digital computer. An explicit approximate solution is, however, shown to give accurate results.**

The analysis can be interpreted directly in terms of mode coupling of the usual rectangular waveguide modes, and the possible extension is indicated to general tensor media and to circular or elliptical waveguide.

Introduction

The millimeter wavelengths are of great interest in applications, such as radar, satellite communication, and radio astronomy, where the requirement is for receivers with the lowest possible noise temperature. Traveling-wave masers are therefore being extended from the centimetric to the millimetric wavelengths [1], [2]. At these wavelengths, heavy loading of the waveguide by suitable paramagnetic material with high permittivity becomes an attractive structure, the high slowing factor and filling factor being achieved without the fabrication difficulties of filter structures. One particular arrangement of interest is rectangular waveguide fully filled with iron-doped rutile, and with the *c*-axis lying in the transverse plane of the waveguide but at 65° to the narrow wall. For proper design of this structure, the dispersion characteristics and field solutions are required of the operating dominant mode. This would be quite trivial if the *c*-axis of the material were aligned with any of the waveguide axes, by means of suitable scaling [3], but this skew anisotropy requires more elaborate solution.

The analysis has been by a variational approach, as the integral form avoids the difficulties of a direct approach (nonuniform convergence of expansions, etc.).

In spite of the considerable attention paid to wave propagation in uniaxial media, and to gyrotropic loaded waveguide, apparently no practical solution has been presented for this particular "skew" loading. It is therefore in order to point out that the following method can be directly extended to rectangular waveguide filled with general tensor permeability or permittivity. For lossy material, a more general variational expression would be required [4], [5], and as the problem is no longer self-adjoint, one would require separate expansions for the "original" and the "adjoint" fields. Again it could be extended directly to circular or elliptical waveguide fully loaded with general anisotropic material. In all cases, the approximate solution reduces to the solution of a matrix eigenvalue problem (the matrix describing the mode coupling between the empty-waveguide modes), and the exact solution is approached as the order of the matrix is increased.

An approximate solution that gives the propagation constant and fields in explicit form is found to be adequate for the particular rutile-filled waveguide.

Analysis

The structure under analysis is given in Fig. 1. The *c*-axis of the uniaxial dielectric lies in the transverse plane (x-y or ξ-η) with the ξ-axis parallel to the *c*-axis. We therefore have

$$\begin{pmatrix} D_\xi \\ D_\eta \\ D_z \end{pmatrix} = \begin{pmatrix} \epsilon_2 & 0 & 0 \\ 0 & \epsilon_1 & 0 \\ 0 & 0 & \epsilon_1 \end{pmatrix} \begin{pmatrix} E_\xi \\ E_\eta \\ E_z \end{pmatrix} \tag{1}$$

and hence

$$\begin{pmatrix} D_x \\ D_y \\ D_z \end{pmatrix} = \begin{pmatrix} \epsilon_{11} & \epsilon_{12} & 0 \\ \epsilon_{12} & \epsilon_{22} & 0 \\ 0 & 0 & \epsilon_1 \end{pmatrix} \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} \tag{2}$$

where

$$\epsilon_{11} = \epsilon_1 \sin^2\theta + \epsilon_2 \cos^2\theta \tag{3}$$

$$\epsilon_{22} = \epsilon_2 \sin^2\theta + \epsilon_1 \cos^2\theta \tag{4}$$

$$\epsilon_{12} = (\epsilon_2 - \epsilon_1)\sin\theta\cos\theta. \tag{5}$$

We use Berk's variational expression [6]

$$\gamma = \frac{\omega \int_s (E^*\epsilon E + H^*\mu H)dS + j\int_s (E^*\cdot\nabla\times H - H^*\cdot\nabla\times E)dS}{\int_s U_z\cdot(E^*\times H - H^*\times E)dS} \tag{6}$$

where the electric and magnetic field vectors are given by $E(x, y)\exp j(\omega t-\gamma z)$ and $H(x, y)\exp j(\omega t-\gamma z)$, and U_z is the z-directed unit vector. For our purposes, μ is a real scalar and ϵ is the real symmetric matrix of (2). The above is a

Manuscript received July 28, 1966; revised December 20, 1966. The work reported in this paper was supported by the Mullard Research Laboratories, Redhill, Surrey, England.

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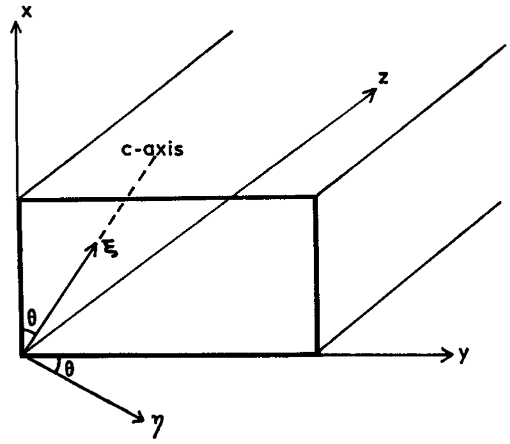

Fig. 1. The waveguide (x, y, z) and uniaxial (ξ, η, z) coordinate systems.

variational expression—provided the trial fields $\boldsymbol{E}$ and $\boldsymbol{H}$ have first derivatives over the cross section, the tangential component of $\boldsymbol{E}$ vanishes over the boundary, and the propagation constant γ is real.

As observed by Gabriel and Brodwin [5], for a loss-free medium we have a self-adjoint system, and so for the trial fields used we are assured of convergence of the Rayleigh-Ritz procedure.

We take as trial fields the following six components:

$$E_x = \sqrt{\frac{\mu}{\epsilon_1}} \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} h_m a_{mn} \cos(m\pi x/x_0) \sin(n\pi y/y_0) \quad (7)$$

$$E_y = \sqrt{\frac{\mu}{\epsilon_1}} \sum_{m=1}^{\infty}\sum_{n=0}^{\infty} h_n b_{mn} \sin(m\pi x/x_0) \cos(n\pi y/y_0) \quad (8)$$

$$E_z = j\sqrt{\frac{\mu}{\epsilon_1}} \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} c_{mn} \sin(m\pi x/x_0) \sin(n\pi y/y_0) \quad (9)$$

$$H_x = \sum_{m=1}^{\infty}\sum_{n=0}^{\infty} h_n d_{mn} \sin(m\pi x/x_0) \cos(n\pi y/y_0) \quad (10)$$

$$H_y = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} h_m e_{mn} \cos(m\pi x/x_0) \sin(n\pi y/y_0) \quad (11)$$

$$H_z = j\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} h_m h_n f_{mn} \cos(m\pi x/x_0) \cos(n\pi y/y_0) \quad (12)$$

where

$$h_m = \begin{cases} 1 & m > 0 \\ 1/\sqrt{2} & m = 0. \end{cases} \quad (13)$$

Clearly these fields are complete and satisfy the boundary restrictions on trial fields. All the coefficients a, b, c, d, e, f can be taken to be real. It can be seen from Maxwell's equations that the particular anisotropy of this problem introduces no coupling between assumed real and imaginary parts of the coefficients.

We now evaluate the integrals of (6), for instance,

$$\int_s \boldsymbol{U}_z \cdot (\boldsymbol{E}^* \times \boldsymbol{H} - \boldsymbol{H}^* \times \boldsymbol{E}) dS$$

$$= 2\sqrt{\frac{\mu}{\epsilon_1}} \int_0^{x_0}\int_0^{y_0} \sum_m \sum_n \sum_{m'} \sum_{n'} \cdot \{h_m a_{mn} h_{m'} e_{m'n'} \cos(m\pi x/x_0)\cos(m'\pi x/x_0) \cdot \sin(n\pi y/y_0)\sin(n'\pi y/y_0) - h_n b_{mn} h_{n'} d_{m'n'} \cdot \sin(m\pi x/x_0)\sin(m'\pi x/x_0)\cos(n\pi y/y_0) \cdot \cos(n'\pi y/y_0)\} dxdy$$

$$= 2\sqrt{\frac{\mu}{\epsilon_1}} \frac{x_0 y_0}{4} \sum_m \sum_n \{a_{mn} e_{mn} - b_{mn} d_{mn}\}. \quad (14)$$

Similarly,

$$\omega \int_s (\boldsymbol{E}^* \epsilon \boldsymbol{E} + \boldsymbol{H}^* \mu \boldsymbol{H}) dS = \omega\mu \frac{x_0 y_0}{4} \sum_m \sum_n \Big\{ d_{mn}^2 + e_{mn}^2 + f_{mn}^2 + \frac{\epsilon_{11}}{\epsilon_1} a_{mn}^2 + \frac{\epsilon_{22}}{\epsilon_1} b_{mn}^2 + c_{mn}^2 + 2\frac{\epsilon_{12}}{\epsilon_1} \sum_{m'} \sum_{n'} a_{mn} b_{m'n'} T(m', m) T(n, n') \Big\} \quad (15)$$

and

$$j\int_s (\boldsymbol{E}^* \cdot \nabla \times \boldsymbol{H} - \boldsymbol{H}^* \cdot \nabla \times \boldsymbol{E}) dS = 2\sqrt{\frac{\mu}{\epsilon_1}} \frac{x_0 y_0}{4} \sum_m \sum_n \cdot \left\{ \frac{n\pi}{y_0} [a_{mn} f_{mn} + c_{mn} d_{mn}] \frac{m\pi}{-x_0} [b_{mn} f_{mn} + c_{mn} e_{mn}] \right\} \quad (16)$$

where

$$T(p, q) = \begin{cases} 0 & \text{if } p - q \text{ is even} \\ \dfrac{4 p h_q}{\pi(p^2 - q^2)} & \text{if } p - q \text{ is odd.} \end{cases} \quad (17)$$

Substituting into the variational expression (6) gives

$$\gamma = \sum_m \sum_n \Big\{ \omega\sqrt{\mu\epsilon_1} \Big[d_{mn}^2 + e_{mn}^2 + f_{mn}^2 + \frac{\epsilon_{11}}{\epsilon_1} a_{mn}^2 + \frac{\epsilon_{22}}{\epsilon_1} b_{mn}^2 + c_{mn}^2 \Big] + 2\Big[\frac{n\pi}{y_0}(a_{mn} f_{mn} + c_{mn} d_{mn}) - \frac{m\pi}{x_0}(b_{mn} f_{mn} + c_{mn} e_{mn}) \Big] + 2\omega\sqrt{\mu\epsilon_1} \frac{\epsilon_{12}}{\epsilon_1} \sum_{m'} \sum_{n'} a_{mn} b_{m'n'} T(m', m) T(n, n') \Big\} \Big/ 2\sum_m \sum_n \{a_{mn} e_{mn} - b_{mn} d_{mn}\}. \quad (18)$$

We now have the propagation constant as the ratio of two real quadratic forms, a form suitable for application of the Rayleigh-Ritz procedure [7], [8].

First Approximation

If we take as the simplest trial solution the fields of the TE_{01} mode in the guide with isotropic filling, we have

$$E_x = \sqrt{\frac{\mu}{\epsilon_1}}\, a_{01} \sin(\pi y/y_0) \tag{19}$$

$$H_y = e_{01} \sin(\pi y/y_0) \tag{20}$$

$$H_z = jf_{01} \cos(\pi y/y_0). \tag{21}$$

Taking a_{01}, e_{01}, f_{01} as the only nonzero parameters in (18) gives

$$\gamma = \left\{\omega\sqrt{\mu\epsilon_1}\left[e_{01}^2 + f_{01}^2 + \frac{\epsilon_{11}}{\epsilon_1} a_{01}^2\right] + \frac{2\pi}{y_0} a_{01} f_{01}\right\} \Big/ 2a_{01}e_{01}. \tag{22}$$

The usual Ritz procedure [7], [8], giving a value of γ stationary with respect to a_{01}, e_{01}, f_{01}, results in

$$\begin{bmatrix} \omega\sqrt{\mu\epsilon_1}\,\epsilon_{11}/\epsilon_1 & -\gamma & \pi/y_0 \\ -\gamma & \omega\sqrt{\mu\epsilon_1} & 0 \\ \pi/y_0 & 0 & \omega\sqrt{\mu\epsilon_1} \end{bmatrix} \begin{bmatrix} a_{01} \\ e_{01} \\ f_{01} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{23}$$

and the determinantal equation reduces to

$$\gamma^2 = \omega^2\mu\epsilon_{11} - (\pi/y_0)^2. \tag{24}$$

These eigenvalues and eigenvectors give us the (approximate) propagation constant and associated fields.

Restricting the trial solution to the fields of the TE_{10} mode, via b_{10}, d_{10}, and f_{10}, similarly leads to

$$\gamma^2 = \omega^2 \mu\epsilon_{22} - (\pi/x_0)^2. \tag{25}$$

Second Approximation

We now extend the Ritz manifold to include the fields corresponding to the TE_{01} and TE_{10} modes of the isotropic guide. Taking $(a_{01}, e_{01}, f_{01}, b_{10}, d_{10}, f_{10})$ as the vector gives our second approximation:

$$\begin{vmatrix} \omega\sqrt{\mu\epsilon_1}\,\epsilon_{11}/\epsilon_1 & -\gamma & \pi/y_0 & \omega\sqrt{\mu\epsilon_1}\,\frac{8}{\pi^2}\frac{\epsilon_{12}}{\epsilon_1} & 0 & 0 \\ -\gamma & \omega\sqrt{\mu\epsilon_1} & 0 & 0 & 0 & 0 \\ \pi/y_0 & 0 & \omega\sqrt{\mu\epsilon_1} & 0 & 0 & 0 \\ \omega\sqrt{\mu\epsilon_1}\,\frac{8}{\pi^2}\frac{\epsilon_{12}}{\epsilon_1} & 0 & 0 & \omega\sqrt{\mu\epsilon_1}\,\epsilon_{22}/\epsilon_1 & \gamma & -\pi/x_0 \\ 0 & 0 & 0 & \gamma & \omega\sqrt{\mu\epsilon_1} & 0 \\ 0 & 0 & 0 & -\pi/x_0 & 0 & \omega\sqrt{\mu\epsilon_1} \end{vmatrix} = 0 \tag{26}$$

or

$$\left\{\gamma^2 - \omega^2\mu\epsilon_{11} + (\pi/y_0)^2\right\}\left\{\gamma^2 - \omega^2\mu\epsilon_{22} + (\pi/x_0)^2\right\} = \left\{\omega^2\mu\epsilon_{12}\frac{8}{\pi^2}\right\}^2. \tag{27}$$

Equation (27) gives us an explicit approximate solution for the propagation constant and, via (26), for the electromagnetic fields.

The matrix elements of (26) involving ϵ_{12} show clearly the coupling between the TE_{10} and TE_{01} modes introduced by the skew anisotropy. Because $\epsilon_{12} = (\epsilon_2 - \epsilon_1)\sin\theta\cos\theta$, we see that the coupling vanishes (as it should) when the anisotropy vanishes, or equally when the c-axis is parallel to either of the waveguide walls.

An Economized Formulation

It should be emphasized that although the field components associated with the TE_{10} and TE_{01} modes have been used above, the ratios of the components have not been constrained to the values for waveguide with isotropic filling. The ratios will generally be different, and to constrain them would generally make the trial field expansions incomplete. On the other hand, it is possible to economize by expressing the longitudinal field components explicitly in terms of the transverse fields.

From Maxwell's equations we have

$$H_z = \frac{j}{\omega\mu}\left\{\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right\} \tag{28}$$

$$E_z = \frac{j}{\omega\epsilon_1}\left\{\frac{\partial H_x}{\partial y} - \frac{\partial H_y}{\partial x}\right\}. \tag{29}$$

As γ is not involved in these equations, we can economize in the number of Ritz parameters simply by substituting for f_{mn} and c_{mn} into (9) and (12) onward, from the following:

$$f_{mn} = \frac{1}{\omega\sqrt{\mu\epsilon_1}}\left\{\frac{m\pi}{x_0} b_{mn} - \frac{n\pi}{y_0} d_{mn}\right\} \tag{30}$$

$$c_{mn} = \frac{1}{\omega\sqrt{\mu\epsilon_1}}\left\{\frac{m\pi}{x_0} e_{mn} - \frac{n\pi}{y_0} d_{mn}\right\}. \tag{31}$$

Now instead of allowing all six field components to be "free," we are restricting the longitudinal components to values that satisfy part of Maxwell's equations. It has been shown that this still leaves "stationary" the resulting expression for γ in terms of the transverse fields.

Substituting from (30) and (31) into (18) gives

$$\gamma = \sum_m \sum_n \Big\{ [\omega^2\mu\epsilon_{11} - (n\pi/y_0)^2] a_{mn}^2 + [\omega^2\mu\epsilon_{22} - (m\pi/x_0)^2] b_{mn}^2 + [\omega^2\mu\epsilon_1 - (n\pi/y_0)^2] d_{mn}^2 + [\omega^2\mu\epsilon_1 - (m\pi/x_0)^2] e_{mn}^2 + 2\frac{m\pi}{x_0}\frac{n\pi}{y_0} a_{mn}b_{mn} + 2\frac{m\pi}{x_0}\frac{n\pi}{y_0} d_{mn}e_{mn} + 2\omega^2\mu\epsilon_{12}\sum_{m'}\sum_{n'} T(m', m)T(n, n')a_{mn}b_{m'n'} \Big\} \Big/ 2\omega\sqrt{\mu\epsilon_1}\sum_m\sum_n \{a_{mn}e_{mn} - b_{mn}d_{mn}\}. \quad (32)$$

This economized version of (18) has the additional advantage of a resulting secular equation which is of the standard matrix eigenvalue form

$$(A - \lambda I)a = 0. \quad (33)$$

This makes for straightforward evaluation by digital computer. Our earlier "first approximation" now corresponds to trial fields with a_{01} and e_{01} as the only nonzero parameters in (32). In place of (22) and (23) we now have

$$\gamma = \{[\omega^2\mu\epsilon_{11} - (\pi/y_0)^2]a_{01}^2 + \omega^2\mu\epsilon_1 e^2_{01}\} / 2\omega\sqrt{\mu\epsilon_1}\, a_{01}e_{01} \quad (34)$$

and

$$\begin{vmatrix} \gamma\omega\sqrt{\mu\epsilon_1} & \omega^2\mu\epsilon_{11} - (\pi/y_0)^2 \\ \omega^2\mu\epsilon_1 & \gamma\omega\sqrt{\mu\epsilon_1} \end{vmatrix} = 0 \quad (35)$$

with the same resulting equation (24).

Our second approximation, with a_{01}, e_{01}, b_{10}, and d_{10} as Ritz parameters, gives

$$\begin{vmatrix} \gamma\,\omega\sqrt{\mu\epsilon_1} & \omega^2\mu\epsilon_1 & 0 & 0 \\ \omega^2\mu\epsilon_{11} - (\pi/y_0)^2 & \gamma\omega\sqrt{\mu\epsilon_1} & -\frac{8}{\pi^2}\omega^2\mu\epsilon_{12} & 0 \\ 0 & 0 & \gamma\omega\sqrt{\mu\epsilon_1} & \omega^2\mu\epsilon_1 \\ \frac{8}{\pi^2}\omega^2\mu\epsilon_{12} & 0 & \omega^2\mu\epsilon_{22} - (\pi/x_0)^2 & \gamma\omega\sqrt{\mu\epsilon_1} \end{vmatrix} = 0 \quad (36)$$

where the determinant has been rearranged to conform to (33). Again, the mode coupling due to the anisotropy between the TE_{01} and TE_{10} can be seen clearly. Equation (36) reduces to the same result as before, in (27).

From our general expression of (32) it can be seen that this particular anisotropy, (2), introduces mode coupling between the usual TE_{mn} and TM_{mn} modes of rectangular waveguide, via the terms involving $\epsilon_{12}T(m', m)T(n, n')$. The solutions will fall into two distinct groups, considered as coupled TE_{mn} and TM_{mn} either a) for all (m, n) with $m+n$ even, or b) for all (m, n) with $m+n$ odd. The families of modes from a) and b) will be quite distinct because the cross-coupling terms $T(m', m)T(n, n')$ of (32) will be zero between the two groups.

Third Approximation

We finally extend the Ritz manifold to include the fields of the TE_{01}, TE_{10}, TE_{12}, and TM_{12} modes of the isotropic loaded guide. Our secular equation is now

$$\begin{vmatrix} \gamma\omega\sqrt{\mu\epsilon_1} & -\omega^2\mu\epsilon_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\omega^2\mu\epsilon_{11} + (\pi/y_0)^2 & \gamma\omega\sqrt{\mu\epsilon_1} & -\frac{8}{\pi^2}\omega^2\mu\epsilon_{12} & 0 & 0 & 0 & \frac{8\sqrt{2}}{3\pi^2}\omega^2\mu\epsilon_{12} & 0 \\ 0 & 0 & \gamma\omega\sqrt{\mu\epsilon_1} & \omega^2\mu\epsilon_1 & 0 & 0 & 0 & 0 \\ \frac{8}{\pi^2}\omega^2\mu\epsilon_{12} & 0 & \omega^2\mu\epsilon_{22} - (\pi/x_0)^2 & \gamma\omega\sqrt{\mu\epsilon_1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \gamma\omega\sqrt{\mu\epsilon_1} & -\omega^2\mu\epsilon_1 + (\pi/x_0)^2 & 0 & -2\pi^2/x_0y_0 \\ 0 & 0 & 0 & 0 & -\omega^2\mu\epsilon_{11} + (2\pi/y_0)^2 & \gamma\omega\sqrt{\mu\epsilon_1} & -2\pi^2/x_0y_0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2\pi^2/x_0y_0 & \gamma\omega\sqrt{\mu\epsilon_1} & \omega^2\mu\epsilon_1 - (2\pi/y_0)^2 \\ -\frac{8\sqrt{2}}{3\pi^2}\omega^2\mu\epsilon_{12} & 0 & 0 & 0 & 2\pi^2/x_0y_0 & 0 & \omega^2\mu\epsilon_{22} - (\pi/x_0)^2 & \gamma\omega\sqrt{\mu\epsilon_1} \end{vmatrix} = 0. \quad (37)$$

Results

Of the three successive approximations given—(24), (27), and (37)—the first two give explicit results for the propagation constant (and the electromagnetic fields). It would be quite straightforward to solve any reasonable order of approximation on a digital computer, by generating the matrix elements from (32) in the computer and solving for the matrix eigenvalue by a standard library program.

Results of the first three approximations are given in Table I and in Fig. 2. They are computed for a waveguide of internal dimensions 0.023 inch by 0.011 inch, filled with iron-doped rutile with $\theta = 65°$, $\epsilon_2/\epsilon_0 = 260$, and $\epsilon_1/\epsilon_0 = 130$.

From these results, we first note the small difference between the second and third approximations. This difference reflects the small effect of the mode coupling of the TE_{12} and TM_{12} to the more dominant TE_{10} and TE_{01} modes. As any higher order of approximation would require the higher-order modes (03, 21, etc.) of the waveguide with isotropic loading, we can safely conclude that, at least for this order of anisotropy (a ratio of 2 to 1), our above approximations are close to the true values. The second approximation is, in fact, sufficiently accurate for all practical purposes.

TABLE I

Guide Wavelength as a Function of Free-Space Wavelength (both in mm)

λ_0	1st Approx.	2nd Approx.	3rd Approx.
1	0.0810	0.0632	0.0629
5	0.4305	0.3650	0.3646
7	0.6463	0.5852	0.5852
8.5	0.8488	0.7922	0.7932
8.6	0.8641	0.8076	0.8088
10	1.1183	1.0577	1.0608
12	1.7368	1.6391	1.6496
14	4.5078	3.7863	3.8970
14.5	$j16.287$	8.4939	9.9296
15	$j4.4047$	$j5.2471$	$j5.0090$

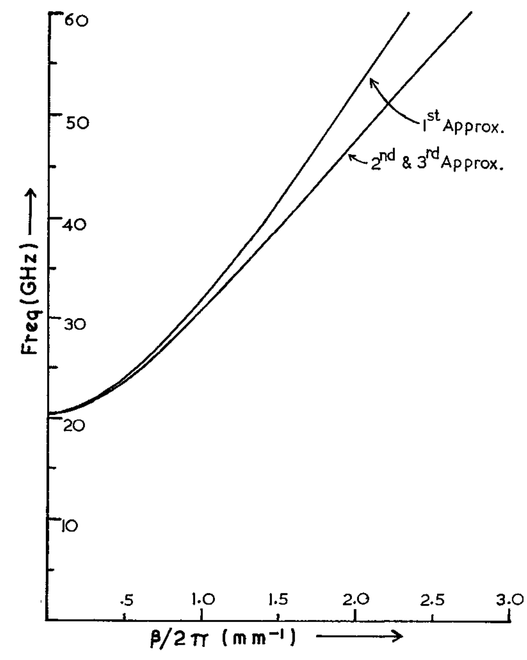

Fig. 2. Dispersion characteristics for waveguide of dimensions 0.023 inch by 0.011 inch, filled with rutile with $\epsilon_2/\epsilon_0 = 260$, $\epsilon_1/\epsilon_0 = 130$, and $\theta = 65°$. The second and third approximations give results indistinguishable on this scale (see Table I).

Modes in the High-Frequency Limit

The value of $\lambda_g = 0.0629$ mm for $\lambda_0 = 1$ mm compares well with 0.0620 mm for TEM wave propagation in the unbounded material with polarization parallel to the *c*-axis. Clearly the electric field of the dominant mode approaches the direction of the *c*-axis in the small wavelength limit, as would be expected with the higher permittivity in that direction. Table I shows, for $\lambda_0 = 1$ mm, the successive approximations 0.0810, 0.0632, and 0.0629 to the dominant waveguide mode that approximates TEM parallel polarization with 0.0620 (all figures being wavelengths in mm). The same computations also give 0.0858 and 0.0868 as approximations to the higher-order waveguide mode that approximates the TEM perpendicular polarization with 0.0877.

Conclusions

The variational method of solution of this problem presents results that can be interpreted directly in terms of coupling of the usual rectangular waveguide modes. For the particular application to the traveling-wave maser, adequate accuracy has been obtained using just the coupled TE_{10} and TE_{01} modes, giving the propagation and electromagnetic fields in explicit form, from (27) and (36). Results of higher accuracy have been obtained by computing the eigenvalues of higher-order matrices.

Solutions have been obtained giving the correct high-frequency limit modes, corresponding to the appropriate two polarizations of TEM waves in the unbounded medium.

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